Short Course on Experimental Dynamic Substructuring

Module #2: General theory

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Short Course Notes For:
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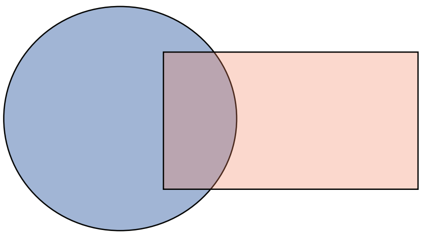
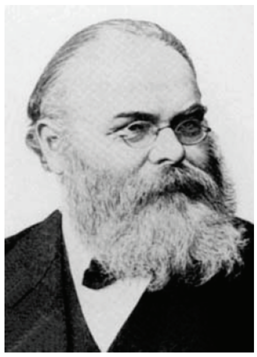
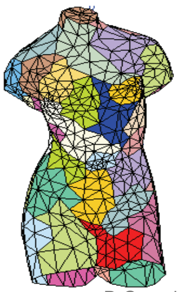
Some history – Divide and Conquer

- The first idea of decoupling and re-assembling to solve a problem was by H. Schwarz, in 1890, who wanted to prove the existence and uniqueness of the solution to the Laplace equation in a complex domain:





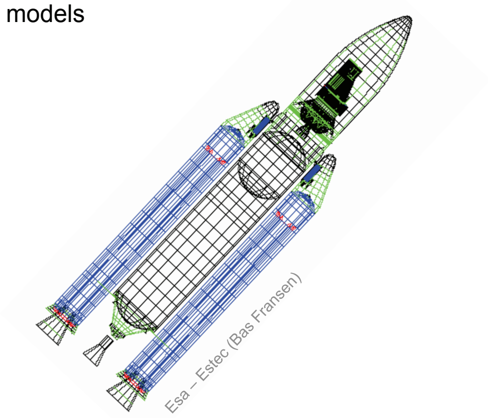
- Later, with the work of Courant (1943), the idea of subdivision was used to define Ritz approximation per elements. Finite Elements were born …



P. Gosselet – ENS Cachan

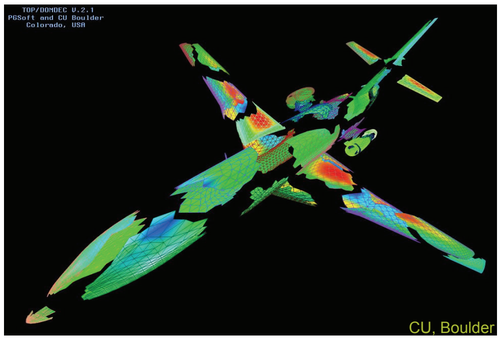
Some history – Divide and Conquer

- In the 60's the idea of partitioning a finite element model in substructures was proposed to reduce the complexity of the models



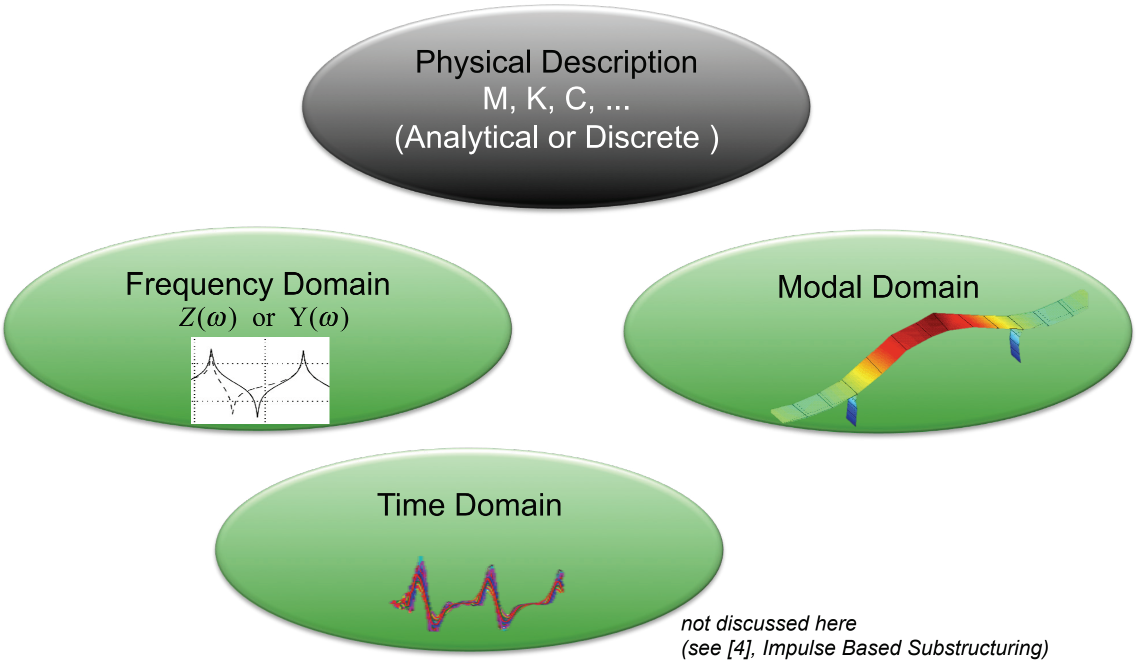
- End of the 60's, but especially in the 80's, decomposing a structure in experimental mechanics was applied in order to simplify the testing. Today new interest thanks to new understanding of methods, better sensors and acquisition.

- In the 80's, in order to speed-up the solution of linear algebraic systems, like in $Ku=F$, the computational domain was cut in sub-domains in order to share work amongst several CPUs (parallel computing). That was the start of *Domain Decomposition.*



Different Domains to apply substructuring [1]

Depending on how the dynamics of the compnents are described ....



not discussed here
(see [4], Impulse Based Substructuring)

Outline

- **Coupling the physical model**
  - 3-field form
  - Primal assembly
  - Dual assembly
  - Summary & Remarks
- **Coupling in the frequency domain**
  - Assembled admittance
  - Example
  - Summary & Remarks
- **Dual coupling in representation space**
- **Weak interface compatibility**
  - By projection
  - By filtering
- **Summary**

References and bibliography

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Coupling the physical model

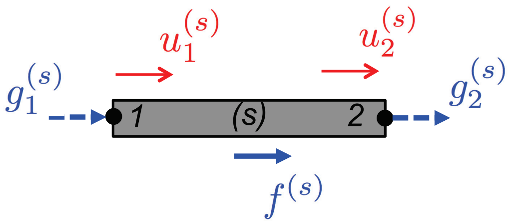
- Dynamic equation of one substructure (s)

link forces on interface

$$\boldsymbol{M}^{(s)}\ddot{\boldsymbol{u}}^{(s)}(t)+\boldsymbol{C}\dot{\boldsymbol{u}}(t)+\boldsymbol{K}^{(s)}\boldsymbol{u}^{(s)}(t)=\boldsymbol{f}^{(s)}(t)+\boldsymbol{g}^{(s)}(t)$$

externally applied forces

- Example of bars.



$$\boldsymbol{K}^{(s)}=k^{(s)}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$\boldsymbol{M}^{(s)}=m^{(s)}\begin{bmatrix}1/2 & 0\\ 0 & 1/2\end{bmatrix}$$

$$\boldsymbol{u}^{(s)}=\begin{bmatrix}u_1^{(s)}\\ u_2^{(s)}\end{bmatrix}$$

$$\boldsymbol{g}^{(s)}=\begin{bmatrix}g_1^{(s)}\\ g_2^{(s)}\end{bmatrix}$$

Coupling the physical model

- ***Block-matrix*** notation for n substructures

$$\boldsymbol{M}^{(s)}\ddot{\boldsymbol{u}}^{(s)}(t)+\boldsymbol{C}\dot{\boldsymbol{u}}(t)+\boldsymbol{K}^{(s)}\boldsymbol{u}^{(s)}(t)=\boldsymbol{f}^{(s)}(t)+\boldsymbol{g}^{(s)}(t)$$

$$\boxed{\boldsymbol{M}\ddot{\boldsymbol{u}}+\boldsymbol{C}\dot{\boldsymbol{u}}+\boldsymbol{K}\boldsymbol{u}=\boldsymbol{f}+\boldsymbol{g}}$$

unassembled system

$$\boldsymbol{M} \triangleq \text{diag}\left(\boldsymbol{M}^{(1)},\ldots,\boldsymbol{M}^{(n)}\right)=\begin{bmatrix}\boldsymbol{M}^{(1)} & . & .\\ . & \ddots & .\\ . & . & \boldsymbol{M}^{(n)}\end{bmatrix}$$

$$\boldsymbol{C} \triangleq \text{diag}\left(\boldsymbol{C}^{(1)},\ldots,\boldsymbol{C}^{(n)}\right)$$

$$\boldsymbol{K} \triangleq \text{diag}\left(\boldsymbol{K}^{(1)},\ldots,\boldsymbol{K}^{(n)}\right)$$

$$\boldsymbol{u} \triangleq \begin{bmatrix}\boldsymbol{u}^{(1)}\\ \vdots\\ \boldsymbol{u}^{(n)}\end{bmatrix},\ \boldsymbol{f} \triangleq \begin{bmatrix}\boldsymbol{f}^{(1)}\\ \vdots\\ \boldsymbol{f}^{(n)}\end{bmatrix},\ \boldsymbol{g} \triangleq \begin{bmatrix}\boldsymbol{g}^{(1)}\\ \vdots\\ \boldsymbol{g}^{(n)}\end{bmatrix}$$

Coupling the physical model: 3-field form

- The system is said to be ***assembled*** when ***2 interface conditions*** are meet:
  - Compatibility of the displacements on the interface

$$\boldsymbol{B}\boldsymbol{u} = \mathbf{0}$$

signed Boolean matrix (0,1,-1)

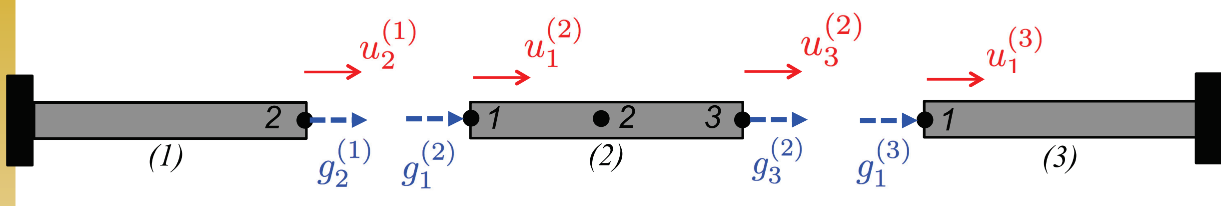
  - Equilibrium of the interface forces (actio-reactio)

$$\boldsymbol{L}^T\boldsymbol{g} = \mathbf{0}$$

Boolean matrix with (0,1)

Coupling the physical model: 3-field form

- Example of bars.



Compatibility on each interface

$$\text{\# interface connections}\left\{\overbrace{\begin{bmatrix}1 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & -1\end{bmatrix}}^{\text{\# unassembled dofs}}\right. \begin{bmatrix}u_2^{(1)}\\ u_1^{(2)}\\ u_2^{(2)}\\ u_3^{(2)}\\ u_1^{(3)}\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\boldsymbol{B}\boldsymbol{u} = \mathbf{0}$$

Computes jump on interface

Equilibrium of internal forces on each assembled dof

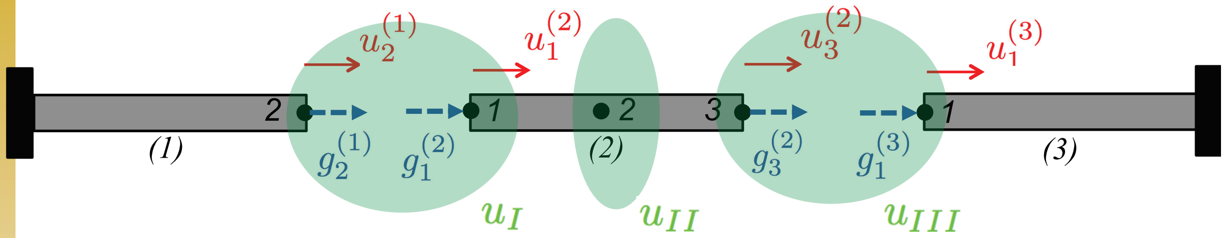
$$\text{\# assembled dof}\left\{\overbrace{\begin{bmatrix}1 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 1\end{bmatrix}}^{\text{\# unassembled dofs}}\right. \begin{bmatrix}g_2^{(1)}\\ g_1^{(2)}\\ g_2^{(2)}\\ g_3^{(2)}\\ g_1^{(3)}\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\boldsymbol{L}^T\boldsymbol{g} = \mathbf{0}$$

Sums up forces on all matching dofs

Coupling the physical model: 3-field form

- Example of bars.



Note: $\boldsymbol{L}$ is also used to define a unique set of assembled dofs

$$\boldsymbol{L}^T\boldsymbol{g} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} g_2^{(1)} \\ g_1^{(2)} \\ g_2^{(2)} \\ g_3^{(2)} \\ g_1^{(3)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_3^{(2)} \\ u_1^{(3)} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_I \\ u_{II} \\ u_{III} \end{bmatrix} = \boldsymbol{L}\boldsymbol{u}_{assembled}$$

and since assembled dofs are per definition compatible:

$$\boldsymbol{B}\boldsymbol{u}_{compatible} = \boldsymbol{B}\boldsymbol{L}\boldsymbol{q} = \mathbf{o} \qquad \forall \boldsymbol{q} \quad \Rightarrow \quad \boxed{\boldsymbol{B}\boldsymbol{L} = \mathbf{o}}$$

$$\boldsymbol{L} = \text{null}(\boldsymbol{B})$$

Coupling the physical model: 3-field form

- Summarizing:
  - an assembled system is described by

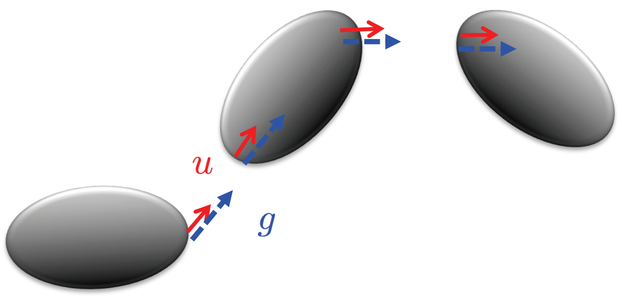
$$\boxed{\begin{cases} \boldsymbol{M}\ddot{\boldsymbol{u}} + \boldsymbol{C}\dot{\boldsymbol{u}} + \boldsymbol{K}\boldsymbol{u} = \boldsymbol{f} + \boldsymbol{g} \\ \boldsymbol{B}\boldsymbol{u} = \mathbf{o} \\ \boldsymbol{L}^T\boldsymbol{g} = \mathbf{o} \end{cases}}$$

→ *local equilibrium*
→ *interface compatibility*
→ *equilibrium of connecting forces*

3-field formulation, using block-matrices of unassembled quantities

  - Additional property

$$\boldsymbol{B}\boldsymbol{L} = \mathbf{o}$$



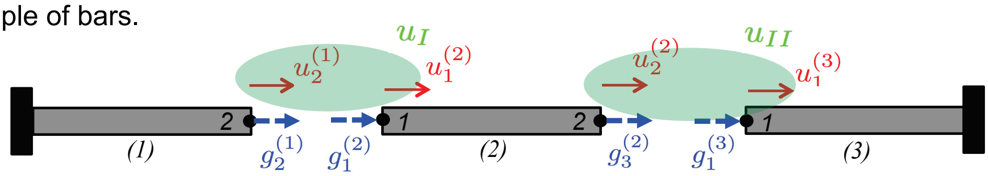
Coupling the physical model: Primal assembly

- Let us now write the assembled problem by considering from the start only a set of **dofs** that are compatible, i.e. originating from a ***common/global assembled*** set

$$\boldsymbol{u}_{glob} \qquad \text{such that} \qquad \boldsymbol{u} = \boldsymbol{L}\boldsymbol{u}_{glob}$$

In other words: the substructure $\boldsymbol{u}^{(s)}$ are picked from a unique global set $\boldsymbol{u}_{glob}$

- Example of bars.



$$\begin{bmatrix} u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_I \\ u_{II} \end{bmatrix} = \boldsymbol{L}\boldsymbol{u}_{glob}$$

Coupling the physical model: Primal assembly

$$\boldsymbol{u}_{glob} \qquad \text{such that} \qquad \boldsymbol{u} = \boldsymbol{L}\boldsymbol{u}_{glob}$$

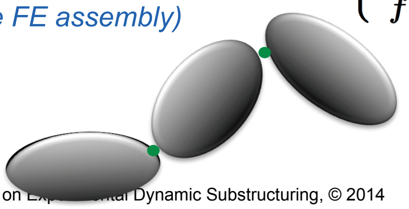
$$\boxed{\begin{cases} \boldsymbol{M}\ddot{\boldsymbol{u}} + \boldsymbol{C}\dot{\boldsymbol{u}} + \boldsymbol{K}\boldsymbol{u} = \boldsymbol{f} + \boldsymbol{g} \\ \boldsymbol{B}\boldsymbol{u} = \boldsymbol{0} \quad \textit{naturally satisfied} \\ \boldsymbol{L}^T\boldsymbol{g} = \boldsymbol{0} \end{cases}} \rightarrow \begin{cases} \boldsymbol{M}\boldsymbol{L}\ddot{\boldsymbol{u}}_{glob} + \boldsymbol{C}\boldsymbol{L}\dot{\boldsymbol{u}}_{glob} + \boldsymbol{K}\boldsymbol{L}\boldsymbol{u}_{glob} = \boldsymbol{f} + \boldsymbol{g} \\ \boldsymbol{L}^T\boldsymbol{g} = \boldsymbol{0} \end{cases}$$

and one can eliminate the unknown interface forces
by assembling the local equilibriums and the interface:

$$\boldsymbol{L}^T\left(\boldsymbol{M}\boldsymbol{L}\ddot{\boldsymbol{u}}_{glob} + \boldsymbol{C}\boldsymbol{L}\dot{\boldsymbol{u}}_{glob} + \boldsymbol{K}\boldsymbol{L}\boldsymbol{u}_{glob} = \boldsymbol{f} + \cancel{\boldsymbol{g}}\right)$$

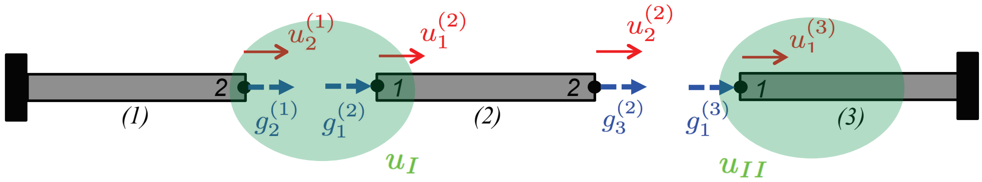
$$\boxed{\tilde{\boldsymbol{M}}\ddot{\boldsymbol{u}}_{glob} + \tilde{\boldsymbol{C}}\dot{\boldsymbol{u}}_{glob} + \tilde{\boldsymbol{K}}\boldsymbol{u}_{glob} = \tilde{\boldsymbol{f}}} \qquad \begin{cases} \tilde{\boldsymbol{M}} & \triangleq & \boldsymbol{L}^T\boldsymbol{M}\boldsymbol{L} \\ \tilde{\boldsymbol{C}} & \triangleq & \boldsymbol{L}^T\boldsymbol{C}\boldsymbol{L} \\ \tilde{\boldsymbol{K}} & \triangleq & \boldsymbol{L}^T\boldsymbol{K}\boldsymbol{L} \\ \tilde{\boldsymbol{f}} & \triangleq & \boldsymbol{L}^T\boldsymbol{f} \end{cases}$$

Primally assembled (like FE assembly)



Coupling the physical model: Primal assembly

- Example of bars.



$$\tilde{\boldsymbol{K}} \triangleq \boldsymbol{L}^T \boldsymbol{K} \boldsymbol{L} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} k^{(1)} & 0 & 0 & 0 \\ 0 & k^{(2)} & -k^{(2)} & 0 \\ 0 & -k^{(2)} & k^{(2)} & 0 \\ 0 & 0 & 0 & k^{(3)} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} k^{(1)} + k^{(2)} & -k^{(2)} \\ -k^{(2)} & k^{(2)} + k^{(3)} \end{bmatrix}$$

exactly like in Finite Elements.

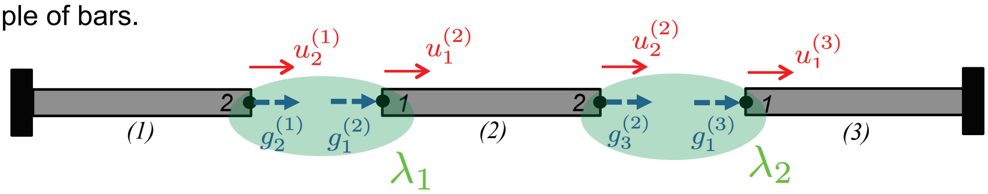
Coupling the physical model: Dual assembly

- Starting again from the 3-field formulation, let us now write the assembled problem by considering from the start only a set of **interface forces** that are in equilibrium, i.e. **equal an opposite**

$$\lambda \quad \text{such that} \quad \boldsymbol{g} = -\boldsymbol{B}^T \lambda$$

In other words: the interface forces are +/– λ on each side of the interface

- Example of bars.



$$\boldsymbol{g} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \boldsymbol{B}^T \lambda$$

Coupling the physical model: Dual assembly

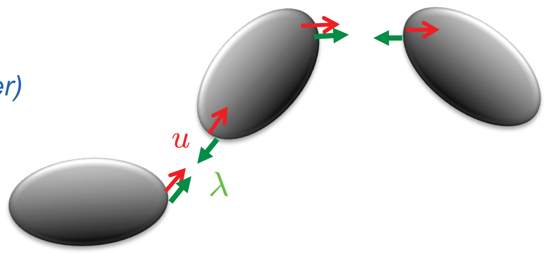
λ such that $\boldsymbol{g} = -\boldsymbol{B}^T\lambda$

$$\left\{\begin{array}{l} \boldsymbol{M}\ddot{\boldsymbol{u}} + \boldsymbol{C}\dot{\boldsymbol{u}} + \boldsymbol{K}\boldsymbol{u} = \boldsymbol{f} + \boldsymbol{g} \\ \boldsymbol{B}\boldsymbol{u} = \mathbf{o} \\ \boldsymbol{L}^T\boldsymbol{g} = \mathbf{o} \text{ naturally satisfied} \end{array}\right. \Rightarrow \left\{\begin{array}{l} \boldsymbol{M}\ddot{\boldsymbol{u}} + \boldsymbol{C}\dot{\boldsymbol{u}} + \boldsymbol{K}\boldsymbol{u} + \boldsymbol{B}^T\lambda = \boldsymbol{f} \\ \boldsymbol{B}\boldsymbol{u} = \mathbf{o} \end{array}\right.$$

and in block matrix form

$$\begin{bmatrix} \boldsymbol{M} & \mathbf{o} \\ \mathbf{o} & \mathbf{o} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{u}} \\ \lambda \end{bmatrix} + \begin{bmatrix} \boldsymbol{C} & \mathbf{o} \\ \mathbf{o} & \mathbf{o} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{u}} \\ \lambda \end{bmatrix} + \begin{bmatrix} \boldsymbol{K} & \boldsymbol{B} \\ \boldsymbol{B}^T & \mathbf{o} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \lambda \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \mathbf{o} \end{bmatrix}$$

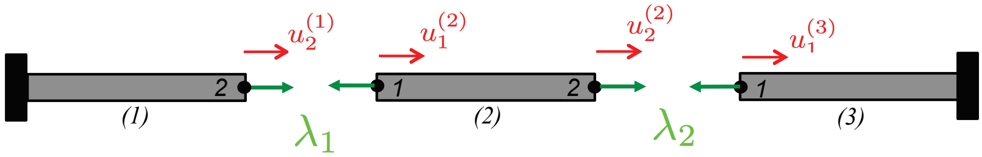
Dually assembled
(like in constrained multibody formulation,
λ = Lagrange multiplier)



Note:
The compatibility condition could also be written on the velocities or accelerations. Careful for drift.

Coupling the physical model: Dual assembly

- Example of bars.



$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{B} \\ \boldsymbol{B}^T & \mathbf{o} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \lambda \end{bmatrix} = \left[\begin{array}{cc} \begin{bmatrix} k^{(1)} & 0 & 0 & 0 \\ 0 & k^{(2)} & -k^{(2)} & 0 \\ 0 & -k^{(2)} & k^{(2)} & 0 \\ 0 & 0 & 0 & k^{(3)} \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \\ \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} & \mathbf{o} \end{array}\right] \begin{bmatrix} u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \\ \lambda_1 \\ \lambda_2 \end{bmatrix}$$

Coupling the physical model: **SUMMARY**

$$\begin{cases} \boldsymbol{M}\ddot{\boldsymbol{u}} + \boldsymbol{C}\dot{\boldsymbol{u}} + \boldsymbol{K}\boldsymbol{u} = \boldsymbol{f} + \boldsymbol{g} \\ \boldsymbol{B}\boldsymbol{u} = \boldsymbol{o} \\ \boldsymbol{L}^T\boldsymbol{g} = \boldsymbol{o} \end{cases}$$

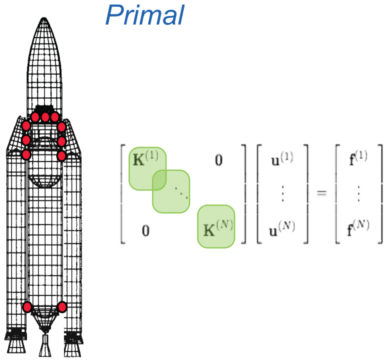
3-field

All mathematically equivalent !

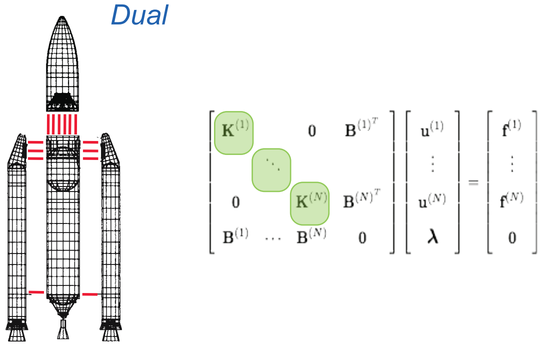
$$\tilde{\boldsymbol{M}}\ddot{\boldsymbol{u}}_{glob} + \tilde{\boldsymbol{C}}\dot{\boldsymbol{u}}_{glob} + \tilde{\boldsymbol{K}}\boldsymbol{u}_{glob} = \tilde{\boldsymbol{f}}$$

$$\begin{bmatrix} \boldsymbol{M} & \boldsymbol{o} \\ \boldsymbol{o} & \boldsymbol{o} \end{bmatrix}\begin{bmatrix} \ddot{\boldsymbol{u}} \\ \lambda \end{bmatrix} + \begin{bmatrix} \boldsymbol{C} & \boldsymbol{o} \\ \boldsymbol{o} & \boldsymbol{o} \end{bmatrix}\begin{bmatrix} \dot{\boldsymbol{u}} \\ \lambda \end{bmatrix} + \begin{bmatrix} \boldsymbol{K} & \boldsymbol{B} \\ \boldsymbol{B}^T & \boldsymbol{o} \end{bmatrix}\begin{bmatrix} \boldsymbol{u} \\ \lambda \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{o} \end{bmatrix}$$

Primal



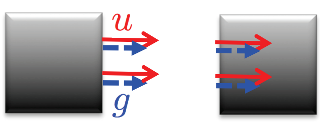
Dual



Coupling the physical model: REMARKS

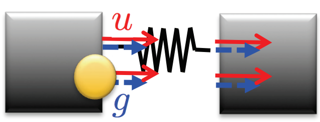
- When enforcing interface equilibrium and compatibility, we have assumed that the interface has no local dynamics:

$$\boldsymbol{B}\boldsymbol{u} = \boldsymbol{o}$$
$$\boldsymbol{L}^T\boldsymbol{g} = \boldsymbol{o}$$



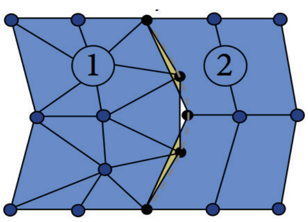
If “simple” interface flexibility, damping or mass is present at the interface, then the interface conditions must be slightly altered

$$\boldsymbol{B}\boldsymbol{u} = \cancel{\boldsymbol{o}}$$
$$\boldsymbol{L}^T\boldsymbol{g} = \cancel{\boldsymbol{o}}$$



and modified primal and dual assembled systems can be derived [2,3]

- If incompatible interface, the dual assembly can still be applied (B non Boolean) e.g. [5]



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Coupling in Frequency Domain

- Exactly the same assembly procedures

$$\begin{cases} \boldsymbol{M}\ddot{\boldsymbol{u}} + \boldsymbol{C}\dot{\boldsymbol{u}} + \boldsymbol{K}\boldsymbol{u} = \boldsymbol{f} + \boldsymbol{g} \\ \boldsymbol{B}\boldsymbol{u} = \boldsymbol{0} \\ \boldsymbol{L}^T\boldsymbol{g} = \boldsymbol{0} \end{cases}$$

$$\boxed{\begin{cases} \boldsymbol{Z}(\omega)\boldsymbol{u}(j\omega) = \boldsymbol{f}(\omega) + \boldsymbol{g}(\omega) \\ \boldsymbol{B}\boldsymbol{u}(\omega) = \boldsymbol{0} \\ \boldsymbol{L}^T\boldsymbol{g}(\omega) = \boldsymbol{0} \end{cases}}$$

3-field

$$\boldsymbol{Z}(j\omega) \triangleq -\omega^2\boldsymbol{M} + j\omega\boldsymbol{C} + \boldsymbol{K}$$

*Structural **impedance** matrix*

$$\boxed{\tilde{\boldsymbol{Z}}\boldsymbol{q} = \tilde{\boldsymbol{f}}}$$

Primal

With the assembled impedance

$$\tilde{\boldsymbol{Z}} \triangleq \boldsymbol{L}^T\boldsymbol{Z}\boldsymbol{L}$$

$$\boxed{\begin{bmatrix} \boldsymbol{Z} & \boldsymbol{B}^T \\ \boldsymbol{B} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{0} \end{bmatrix}}$$

where $\boldsymbol{Z}$ is the block-matrix of impedances of each individual substructure (i.e. free interface)

Commonly called
Frequency Based Substructuring or FBS

Coupling in Frequency Domain: Assembled Admittance

The dual assembly is usually preferred when the substructure data are coming from measurements. Why ?

$$\boxed{\begin{bmatrix} \boldsymbol{Z} & \boldsymbol{B}^T \\ \boldsymbol{B} & \mathbf{o} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \lambda \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \mathbf{o} \end{bmatrix}} \Rightarrow \boldsymbol{u} = \boldsymbol{Y}\left(\boldsymbol{f} - \boldsymbol{B}^T\lambda\right)$$

where

$$\boldsymbol{Y}(j\omega) \triangleq \boldsymbol{Z}^{-1} = \left(-\omega^2\boldsymbol{M} + j\omega\boldsymbol{C} + \boldsymbol{K}\right)$$

Block-matrix of substructure ***admittances***

$$\boldsymbol{B}\left(\boldsymbol{Y}\boldsymbol{f} - \boldsymbol{Y}\boldsymbol{B}^T\lambda\right) = \mathbf{o}$$

$$\boxed{(\boldsymbol{B}\boldsymbol{Y}\boldsymbol{B})\lambda = \boldsymbol{B}\boldsymbol{Y}\boldsymbol{f}}$$

Dual interface problem:
"how much interface forces is needed to close the interface gap resulting from external forces"

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{Y}\boldsymbol{f} - \boldsymbol{Y}\boldsymbol{B}^T(\boldsymbol{B}\boldsymbol{Y}\boldsymbol{B}^T)^{-1}\boldsymbol{B}\boldsymbol{Y}\boldsymbol{f} \\ &= \left(\boldsymbol{Y} - \boldsymbol{Y}\boldsymbol{B}^T(\boldsymbol{B}\boldsymbol{Y}\boldsymbol{B}^T)^{-1}\boldsymbol{B}\boldsymbol{Y}\right)\boldsymbol{f} \end{aligned}$$

=Assembled Admittance

Coupling in Frequency Domain: Assembled Admittance

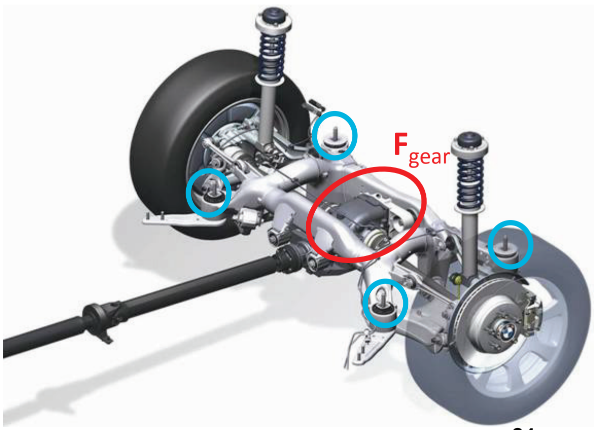
Summarizing:

If one knows the admittance of each substructure free on its interface, then the dynamic response of the assembled system is

$$\boxed{\boldsymbol{u} = \left(\boldsymbol{Y} - \boldsymbol{Y}\boldsymbol{B}^T(\boldsymbol{B}\boldsymbol{Y}\boldsymbol{B}^T)^{-1}\boldsymbol{B}\boldsymbol{Y}\right)\boldsymbol{f}}$$

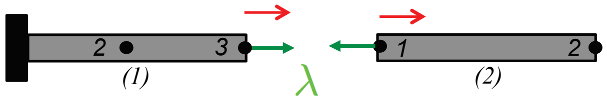
i.e. *one can computed the admittance of the assembled system using local admittances* !

→ useful in experimental dynamics where admittances of non-connected components are measured.



Coupling in Frequency Domain: Example

- Example of bars.



$$\boldsymbol{u} = \left(\boldsymbol{Y} - \boldsymbol{Y}\boldsymbol{B}^{T}\left(\boldsymbol{B}\boldsymbol{Y}\boldsymbol{B}^{T}\right)^{-1}\boldsymbol{B}\boldsymbol{Y}\right)\boldsymbol{f}$$

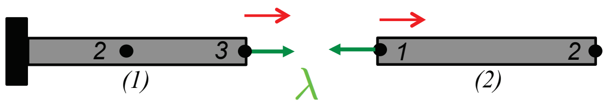
$$\boldsymbol{Y} = \begin{bmatrix} Y_{22}^{(1)} & Y_{23}^{(1)} & 0 & 0 \\ Y_{32}^{(1)} & Y_{33}^{(1)} & 0 & 0 \\ 0 & 0 & Y_{11}^{(2)} & Y_{12}^{(2)} \\ 0 & 0 & Y_{21}^{(2)} & Y_{22}^{(2)} \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix} \qquad \boldsymbol{u} = \begin{bmatrix} u_2^{(1)} \\ u_3^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \end{bmatrix}$$

assembled
interface flexibility

$$\begin{bmatrix} u_2^{(1)} \\ u_3^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \end{bmatrix} = \left\{ \begin{bmatrix} Y_{22}^{(1)} & Y_{23}^{(1)} & 0 & 0 \\ Y_{32}^{(1)} & Y_{33}^{(1)} & 0 & 0 \\ 0 & 0 & Y_{11}^{(2)} & Y_{12}^{(2)} \\ 0 & 0 & Y_{21}^{(2)} & Y_{22}^{(2)} \end{bmatrix} - \begin{bmatrix} Y_{23}^{(1)} \\ Y_{33}^{(1)} \\ -Y_{11}^{(2)} \\ -Y_{21}^{(2)} \end{bmatrix} \left(Y_{33}^{(1)} + Y_{11}^{(2)}\right)^{-1} \begin{bmatrix} Y_{32}^{(1)} & Y_{33}^{(1)} & -Y_{11}^{(2)} & -Y_{12}^{(2)} \end{bmatrix} \right\} \begin{bmatrix} f_2^{(1)} \\ f_3^{(1)} \\ f_1^{(2)} \\ f_2^{(2)} \end{bmatrix}$$

Coupling in Frequency Domain: Example

- Example of bars.



$$\boldsymbol{u} = \left(\boldsymbol{Y} - \boldsymbol{Y}\boldsymbol{B}^{T}\left(\boldsymbol{B}\boldsymbol{Y}\boldsymbol{B}^{T}\right)^{-1}\boldsymbol{B}\boldsymbol{Y}\right)\boldsymbol{f}$$

$$\boldsymbol{Y} = \begin{bmatrix} Y_{22}^{(1)} & Y_{23}^{(1)} & 0 & 0 \\ Y_{32}^{(1)} & Y_{33}^{(1)} & 0 & 0 \\ 0 & 0 & Y_{11}^{(2)} & Y_{12}^{(2)} \\ 0 & 0 & Y_{21}^{(2)} & Y_{22}^{(2)} \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix} \qquad \boldsymbol{u} = \begin{bmatrix} u_2^{(1)} \\ u_3^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \end{bmatrix}$$

incompatibility between substructures
when no connection

$$\begin{bmatrix} u_2^{(1)} \\ u_3^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \end{bmatrix} = \left\{ \begin{bmatrix} Y_{22}^{(1)} & Y_{23}^{(1)} & 0 & 0 \\ Y_{32}^{(1)} & Y_{33}^{(1)} & 0 & 0 \\ 0 & 0 & Y_{11}^{(2)} & Y_{12}^{(2)} \\ 0 & 0 & Y_{21}^{(2)} & Y_{22}^{(2)} \end{bmatrix} - \begin{bmatrix} Y_{23}^{(1)} \\ Y_{33}^{(1)} \\ -Y_{11}^{(2)} \\ -Y_{21}^{(2)} \end{bmatrix} \left(Y_{33}^{(1)} + Y_{11}^{(2)}\right)^{-1} \begin{bmatrix} Y_{32}^{(1)} & Y_{33}^{(1)} & -Y_{11}^{(2)} & -Y_{12}^{(2)} \end{bmatrix} \right\} \begin{bmatrix} f_2^{(1)} \\ f_3^{(1)} \\ f_1^{(2)} \\ f_2^{(2)} \end{bmatrix}$$

interface force λ needed for compatibility

response to applied forces

response to interface forces

Coupling in Frequency Domain: Summary

Summary to remember !

$$\boxed{u = \left(Y - YB^T(BYB^T)^{-1}BY\right)f}$$

where Y contains in its blocks the uncoupled admittances

where B is signed Boolean and describes the connection topology

$$\boxed{u = \left(Y - YB^T(BYB^T)^{-1}BY\right)f}$$

incompatibility between substructures when no connection

interface force λ needed for compatibility

response to applied forces

response to interface forces

Coupling in Frequency Domain: Remarks

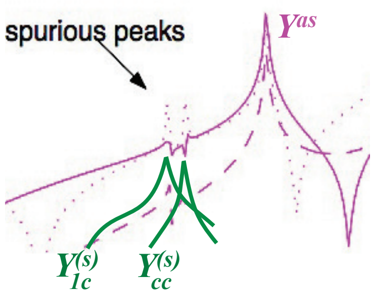
$$\boxed{u = \left(Y - YB^T(BYB^T)^{-1}BY\right)f}$$

- + Systematic assembly for any number of substructures and any topology
- - needs admittances for all interface dofs …
  - - not always possible to place sensors on interface
  - - not always possible to measure **all** dofs on interface
- - implies *inverting* the assembled interface flexibility

  → explosion of measurement errors [6]

  e.g. measurement noise

  e.g. poles in $Y^{(s)}$ not identical for all coefficient [7]



Coupling in Frequency Domain: Remarks

$$\boxed{u = \left(Y - YB^T(BYB^T)^{-1}BY\right)f}$$

- \+ Systematic assembly for any number of substructures and any topology
- \- needs admittances for all interface dofs …
  - \- not always possible to place sensors on interface
  - \- not always possible to measure **all** dofs on interface

expansion of measurements in the vicinity of interface using local shapes (e.g. Serep). See an application in MOD09.

- \- implies *inverting* the assembled interface flexibility
  → explosion of measurement errors [6]
  e.g. measurement noise
  e.g. poles in $Y^{(s)}$ not identical for all coefficient [7]

- SVD filtering of data, e.g. [8]
- Modal synthesis of data (if low modal density)
- Correction with FE model [9]
- Cleaning Y^{as} *a posteriori* [10]
- Weakening of interface compatibility → next sections

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Dual coupling in representation space (Modal)

- How do the assembled equations look like if

  each substructure is approximated by a
  reduced set of (modal) shapes stored in the columns of $\boldsymbol{R}^{(s)}$

$$\boldsymbol{u}^{(s)} \simeq \boldsymbol{R}^{(s)}\eta^{(s)}$$

in local equilibria

$$\boldsymbol{M}^{(s)}\ddot{\boldsymbol{u}}^{(s)}(t) + \boldsymbol{C}\dot{\boldsymbol{u}}(t) + \boldsymbol{K}^{(s)}\boldsymbol{u}^{(s)}(t) = \boldsymbol{f}^{(s)}(t) + \boldsymbol{g}^{(s)}(t)$$

$$\boldsymbol{M}^{(s)}\boldsymbol{R}^{(s)}\ddot{\eta}^{(s)} + \boldsymbol{C}^{(s)}\boldsymbol{R}^{(s)}\dot{\eta}^{(s)} + \boldsymbol{K}^{(s)}\boldsymbol{R}^{(s)}\eta^{(s)} = \boldsymbol{f}^{(s)} + \boldsymbol{g}^{(s)} + \boldsymbol{r}^{(s)}$$

Equilibrium error due to response approximation

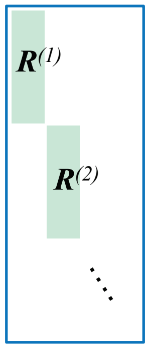
or in block notations

$$\begin{aligned} \boldsymbol{u} &\simeq \boldsymbol{R}\eta \\ \boldsymbol{R} &\triangleq \text{diag}\left(\boldsymbol{R}^{(1)}, \ldots, \boldsymbol{R}^{(n)} \right) \end{aligned}$$

in local equilibria

$$\boldsymbol{M}\ddot{\boldsymbol{u}} + \boldsymbol{C}\dot{\boldsymbol{u}} + \boldsymbol{K}\boldsymbol{u} = \boldsymbol{f} + \boldsymbol{g}$$

$$\boldsymbol{M}\boldsymbol{R}\ddot{\eta} + \boldsymbol{C}\boldsymbol{R}\dot{\eta} + \boldsymbol{K}\boldsymbol{R}\eta = \boldsymbol{f} + \boldsymbol{g} + \boldsymbol{r}$$



Dual coupling in representation space (Modal)

$$\boldsymbol{u} \simeq \boldsymbol{R}\eta \qquad \boldsymbol{M}\boldsymbol{R}\ddot{\eta} + \boldsymbol{C}\boldsymbol{R}\dot{\eta} + \boldsymbol{K}\boldsymbol{R}\eta = \boldsymbol{f} + \boldsymbol{g} + \boldsymbol{r}$$

Equilibrium error due to response approximation

Asking that the residual force creates no work in the approximation space
(as done in reduction techniques – virtual work),

$$\boldsymbol{R}^T\left(\boldsymbol{M}\boldsymbol{R}\ddot{\eta} + \boldsymbol{C}\boldsymbol{R}\dot{\eta} + \boldsymbol{K}\boldsymbol{R}\eta = \boldsymbol{f} + \boldsymbol{g} + \cancel{\boldsymbol{r}}\right)$$

$$\rightarrow \quad \boxed{\boldsymbol{M}_m\ddot{\eta} + \boldsymbol{C}_m\dot{\eta} + \boldsymbol{K}_m\eta = \boldsymbol{f}_m + \boldsymbol{g}_m}$$

where
$$\left\{ \begin{aligned} \boldsymbol{M}_m &\triangleq \boldsymbol{R}^T\boldsymbol{M}\boldsymbol{R} \\ \boldsymbol{C}_m &\triangleq \boldsymbol{R}^T\boldsymbol{C}\boldsymbol{R} \\ \boldsymbol{K}_m &\triangleq \boldsymbol{R}^T\boldsymbol{K}\boldsymbol{R} \\ \boldsymbol{f}_m &\triangleq \boldsymbol{R}^T\boldsymbol{f} \\ \boldsymbol{g}_m &\triangleq \boldsymbol{R}^T\boldsymbol{g} \end{aligned} \right.$$
are the matrices in the representation space (reduced)

The compatibility condition $\boldsymbol{B}\boldsymbol{u} = \boldsymbol{0}$ transforms into $\boxed{\boldsymbol{B}_m\eta = \boldsymbol{0}}$

where $\boxed{\boldsymbol{B}_m \triangleq \boldsymbol{B}\boldsymbol{R}}$

Dual coupling in representation space (Modal)

Following the same reasoning as for the general case, one can write the dually assembled problem in the representation space [1]

$$\begin{bmatrix} \boldsymbol{M}_m & \mathbf{o} \\ \mathbf{o} & \mathbf{o} \end{bmatrix} \begin{bmatrix} \ddot{\eta} \\ \lambda \end{bmatrix} + \begin{bmatrix} \boldsymbol{C}_m & \mathbf{o} \\ \mathbf{o} & \mathbf{o} \end{bmatrix} \begin{bmatrix} \dot{\eta} \\ \lambda \end{bmatrix} + \begin{bmatrix} \boldsymbol{K}_m & \boldsymbol{B}_m \\ \boldsymbol{B}_m^T & \mathbf{o} \end{bmatrix} \begin{bmatrix} \eta \\ \lambda \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_m \\ \mathbf{o} \end{bmatrix}$$

REMARKS:

- When expressed in the frequency domain, the assembled problem in representation space is

$$\begin{bmatrix} \boldsymbol{Z}_m & \boldsymbol{B}_m{}^T \\ \boldsymbol{B}_m & \mathbf{o} \end{bmatrix} \begin{bmatrix} \eta \\ \lambda \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_m \\ \mathbf{o} \end{bmatrix} \quad \text{where} \quad \boldsymbol{Z}_m = \boldsymbol{R}^T \boldsymbol{Z} \boldsymbol{R}$$

- Looks very similar to the Dual problem in the physical space but …
  - … the matrices are reduced and expressing the physical properties in the representation space
  - … the unknowns are now the amplitudes of the representation vectors

- It is important to build the reduction space $\boldsymbol{R}$ on shapes that can properly represent the response of the *assembled* system
  → some methods use modes of the system when assembled with a dummy mimicing roughly neighboring structures (see the transmission simulator idea in MOD9)

Dual coupling in representation space (Modal)

$$\begin{bmatrix} \boldsymbol{M}_m & \mathbf{o} \\ \mathbf{o} & \mathbf{o} \end{bmatrix} \begin{bmatrix} \ddot{\eta} \\ \lambda \end{bmatrix} + \begin{bmatrix} \boldsymbol{C}_m & \mathbf{o} \\ \mathbf{o} & \mathbf{o} \end{bmatrix} \begin{bmatrix} \dot{\eta} \\ \lambda \end{bmatrix} + \begin{bmatrix} \boldsymbol{K}_m & \boldsymbol{B}_m \\ \boldsymbol{B}_m^T & \mathbf{o} \end{bmatrix} \begin{bmatrix} \eta \\ \lambda \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_m \\ \mathbf{o} \end{bmatrix}$$

REMARKS (continued):

- If the approximation space is the modal space of the substructures (free interface),

$$\boldsymbol{u}^{(s)} = \boldsymbol{\Phi}^{(s)} \eta^{(s)} \quad \text{where} \quad \boldsymbol{K}^{(s)} \boldsymbol{\Phi}^{(s)} = \boldsymbol{M}^{(s)} \boldsymbol{\Phi}^{(s)} \boldsymbol{\Omega}^{(s)^2}$$

then in frequency domain (with proportional damping)

$$\begin{bmatrix} \begin{bmatrix} -\omega^2 \boldsymbol{I} + i\omega 2\zeta^{(1)} \boldsymbol{\Omega}^{(1)} + \boldsymbol{\Omega}^{(1)^2} & 0 \\ 0 & \ddots \end{bmatrix} & \begin{bmatrix} \boldsymbol{\Phi}^{(1)^T} \boldsymbol{B}^{(1)^T} \\ \vdots \end{bmatrix} \\ \begin{bmatrix} \boldsymbol{B}^{(1)} \boldsymbol{\Phi}^{(1)} & \cdots \end{bmatrix} & 0 \end{bmatrix} \begin{bmatrix} \eta^{(1)} \\ \vdots \\ \lambda \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}^{(1)^T} \boldsymbol{f}^{(1)} \\ \vdots \\ 0 \end{bmatrix}$$

- One major issue: the condition $\boldsymbol{B}\boldsymbol{R}\eta = \boldsymbol{B}_m \eta = \mathbf{o}$ requires full compatibility on the interface. Since each substructures is represented by a small numbers of local modes, this might lead to ***locking*** (the compatibility can be satisfied only when 0 response)
  → ***need for weakening the interface copatibility***

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Weak interface compatibility

Enforcing the strong compatibility for the entire interface can lead to serious problems when (see earlier discussions) ….

- the admittance contains measurement errors and the interface is stiff
- when the substructures are approximated by a limited number of representative shapes that do not necessarily match on the interface (locking)

→ 2 different techniques to relax the interface compatibility (***weakening***)

1. requiring that only a projection of the interface gap is 0
   → **Projected compatibility**
2. projecting the interface displacements on each side of the interface on a common reduced space and enforcing compatibility only for that space
   → **Filtering**

Weak interface compatibility: Projected Compatibility

Starting from the frequency representation with physical dofs (would be similar for modal representation):

$$\boxed{\begin{bmatrix} \boldsymbol{Z} & \boldsymbol{B}^T \\ \boldsymbol{B} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \lambda \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \mathbf{0} \end{bmatrix}}$$

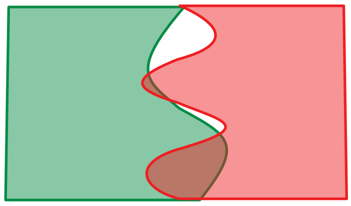
In order to weaken the compatibility condition, we will require only that

the projection of the interface gap on a small number of interface modes is 0

$$\boxed{\boldsymbol{\Gamma}_I^T(\boldsymbol{B}\boldsymbol{u}) = 0}$$

where $\boldsymbol{\Gamma}_I$ contains the interface projection shapes.

For instance if $\boldsymbol{\Gamma}_I = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$, ….



….. one imposes only an average compatibility

Weak interface compatibility: Projected Compatibility

One can show that the dual problem then writes

$$\begin{bmatrix} \boldsymbol{Z} & \boldsymbol{B}^T\boldsymbol{\Gamma}_I \\ \boldsymbol{\Gamma}_I^T\boldsymbol{B} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \lambda_\Gamma \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \mathbf{0} \end{bmatrix}$$

- small number of compatibility conditions on the interface (projected)
- small number of interface force variables λ_Γ
- can be interpreted as if the compatibility is enforces through interface force that are built on the interface projection shapes: $\lambda \simeq \Gamma_I \lambda_\Gamma$
- Similarly, if the substructure displacements are approximated in a representation space:

$$\boldsymbol{u} \simeq \boldsymbol{R}\eta \quad \text{and} \quad \begin{bmatrix} \boldsymbol{Z}_m & \boldsymbol{B}_m{}^T\boldsymbol{\Gamma}_I \\ \boldsymbol{\Gamma}_I^T\boldsymbol{B}_m & \mathbf{0} \end{bmatrix} \begin{bmatrix} \eta \\ \lambda_\Gamma \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_m \\ \mathbf{0} \end{bmatrix}$$

in which the displacements ***and*** the forces are approximated.

Weak interface compatibility: Filtering

One fits the interface displacement on both sides of the interface on a common displacement shape.

Calling $\boldsymbol{\Phi}_\Gamma$ the assumed modes for the interface displacements,

$$\boldsymbol{u}^{(s)} = \begin{bmatrix} \boldsymbol{u}_i^{(s)} \\ \boldsymbol{u}_c^{(s)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}_i^{(s)} \\ \boldsymbol{\Phi}_\Gamma \alpha^{(s)} + \tilde{\boldsymbol{u}}_c^{(s)} \end{bmatrix}$$

internal dofs

connecting dofs

part not representable in $\boldsymbol{\Phi}_\Gamma$

common interface modes

One will enforce a weak compatibility by asking only that

the amplitudes of $\boldsymbol{\Phi}_\Gamma$ should be equal on each side of the interface: $\boldsymbol{B}\alpha = 0$

$$\boldsymbol{u}_c^{(s)} \simeq \boldsymbol{\Phi}_\Gamma \alpha^{(s)}$$

and by least square

$$\alpha^{(s)} = \left(\boldsymbol{\Phi}_\Gamma^T \boldsymbol{\Phi}_\Gamma\right)^{-1} \boldsymbol{\Phi}_\Gamma^T \boldsymbol{u}_c^{(s)}$$

$$\Rightarrow \quad \boldsymbol{B}\left(\boldsymbol{\Phi}_\Gamma^T \boldsymbol{\Phi}_\Gamma\right)^{-1} \boldsymbol{\Phi}_\Gamma^T \boldsymbol{u}_c = \boldsymbol{B}\boldsymbol{F}\boldsymbol{u} = 0$$

filter that removes everything that is not in the space of $\boldsymbol{\Phi}_\Gamma$

Weak interface compatibility: Filtering

Finally the dual problem writes

$$\begin{bmatrix} \boldsymbol{Z} & \boldsymbol{F}^T \boldsymbol{B}^T \\ \boldsymbol{B}\boldsymbol{F} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \lambda_\alpha \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{0} \end{bmatrix}$$

- similarly if the displacements are approximated in a representation space

$$\begin{bmatrix} \boldsymbol{Z}_m & \boldsymbol{R}^T \boldsymbol{F}^T \boldsymbol{B}^T \\ \boldsymbol{B}\boldsymbol{F}\boldsymbol{R} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \eta \\ \lambda_\alpha \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_m \\ \boldsymbol{0} \end{bmatrix}$$

- λ_α has the meaning of generalized forces associated to the interface displacements
- An important application: when $\boldsymbol{\Phi}_\Gamma$ contains the rigid body motion of the interface, then λ_α are the 6 force and moment resultants at the interface
  - → often used for rigid interfaces (bolts, brackets ….), allowing to construct rotational dofs
  - → in that case the method is also called EMPC (Explicit Multi-Point connection) [11,12]

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Summary

$$\boxed{\begin{bmatrix} \boldsymbol{Z} & \boldsymbol{B}^T \\ \boldsymbol{B} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \lambda \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \mathbf{0} \end{bmatrix}} \Leftrightarrow \boxed{\boldsymbol{u} = \overbrace{\left(\boldsymbol{Y} - \boldsymbol{Y}\boldsymbol{B}^T(\boldsymbol{B}\boldsymbol{Y}\boldsymbol{B}^T)^{-1}\boldsymbol{B}\boldsymbol{Y}\right)}^{\text{assembled admittance}}\boldsymbol{f}}$$

or

$$\begin{bmatrix} \boldsymbol{Z}_m & \boldsymbol{B}_m{}^T \\ \boldsymbol{B}_m & \mathbf{0} \end{bmatrix} \begin{bmatrix} \eta \\ \lambda \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_m \\ \mathbf{0} \end{bmatrix}$$

in reduced representation space $\boldsymbol{u} \simeq \boldsymbol{R}\eta$

$$\boldsymbol{Z}_m = \boldsymbol{R}^T \boldsymbol{Z} \boldsymbol{R}$$

Weakening of compatibility by *projection*:

$$\lambda \simeq \boldsymbol{\Gamma}_I \lambda_\Gamma$$

$$\begin{bmatrix} \boldsymbol{Z} & \boldsymbol{B}^T\boldsymbol{\Gamma}_I \\ \boldsymbol{\Gamma}_I^T\boldsymbol{B} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \lambda_\Gamma \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \mathbf{0} \end{bmatrix}$$

or

$$\begin{bmatrix} \boldsymbol{Z}_m & \boldsymbol{B}_m{}^T\boldsymbol{\Gamma}_I \\ \boldsymbol{\Gamma}_I^T\boldsymbol{B}_m & \mathbf{0} \end{bmatrix} \begin{bmatrix} \eta \\ \lambda_\Gamma \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_m \\ \mathbf{0} \end{bmatrix}$$

Weakening of compatibility by *filtering*:

$$\boldsymbol{u}_c^{(s)} \simeq \boldsymbol{\Phi}_\Gamma \alpha^{(s)}$$
$$\boldsymbol{F} = \left(\boldsymbol{\Phi}_\Gamma^T \boldsymbol{\Phi}_\Gamma\right)^{-1} \boldsymbol{\Phi}_\Gamma^T$$

$$\begin{bmatrix} \boldsymbol{Z} & \boldsymbol{F}^T\boldsymbol{B}^T \\ \boldsymbol{B}\boldsymbol{F} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \lambda_\alpha \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \mathbf{0} \end{bmatrix}$$

or

$$\begin{bmatrix} \boldsymbol{Z}_m & \boldsymbol{R}^T\boldsymbol{F}^T\boldsymbol{B}^T \\ \boldsymbol{B}\boldsymbol{F}\boldsymbol{R} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \eta \\ \lambda_\alpha \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_m \\ \mathbf{0} \end{bmatrix}$$

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